Example: Reparamatorize $\vec{r}(t) = \langle 3\sin(t), 2t, 3\cos(t) \rangle$ by arc length measured from $t=0$

Sol: First we compute the Arc length function

$$S(q) = \int_{q=0}^{t} |r'(q)|\, dq \qquad r'(t) = \langle 3\cos(t), 2, -3\sin(t) \rangle$$

$$|r'(t)| = \sqrt{9\cos^2(t) + 4 + -9\sin^2(t)} = \sqrt{9+4} = \sqrt{13}$$

$$= \int_{q=0}^{t} \sqrt{13}\, dq = \sqrt{13}\, q \Big|_0^t = \sqrt{13}t - \sqrt{13}(0)$$

$$S(q) = \sqrt{13}\, t$$

$$t = \frac{s}{\sqrt{13}}$$

Finally, our reparamitorized function is

$$\vec{p}(s) = \vec{r}(t(s)) = \left\langle 3\sin\left(\frac{s}{\sqrt{13}}\right), \frac{2s}{\sqrt{13}}, -3\cos\left(\frac{s}{\sqrt{13}}\right) \right\rangle$$

NB: For the example above

$$\vec{p}'(s) = \left\langle \frac{3}{\sqrt{13}}\cos\left(\frac{s}{\sqrt{13}}\right), \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\sin\left(\frac{s}{\sqrt{13}}\right) \right\rangle$$

$$|p'(s)| = \sqrt{\left(\frac{3}{\sqrt{13}}\cos\left(\frac{s}{\sqrt{13}}\right)\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2 + \left(-\frac{3}{\sqrt{13}}\sin\left(\frac{s}{\sqrt{13}}\right)\right)^2}$$

$$= \sqrt{\frac{9}{13}\cos^2\left(\frac{s}{\sqrt{13}}\right) + \frac{4}{13} + \frac{9}{13}\sin^2\left(\frac{s}{\sqrt{13}}\right)}$$

$$= \sqrt{\frac{9}{13}\left(\cos^2\left(\frac{s}{\sqrt{13}}\right) + \sin^2\left(\frac{s}{\sqrt{13}}\right)\right) + \frac{4}{13}}$$

$$= \sqrt{\frac{9}{13} + \frac{4}{13}} = \sqrt{\frac{13}{13}} = 1 \quad \text{for all } S!$$

Hence, this reparamitorized curve has unit speed

In general, a curve paramitorized by arc length always has unit speed

Some example problems

Ex: Find the velocity and acceleration of $\vec{r}(t) = \langle 2^t, t^2, \ln(t+1) \rangle$ at $t=1$

$= \langle e^{\ln(2t)}, t^2, \ln(t+1) \rangle$

Sol: $\vec{v}(t) = r'(t) = \langle \ln(2) e^{\ln(2)t}, 2t, \frac{1}{t+1} \rangle = \langle \ln(2) \cdot 2^t, 2t, \frac{1}{t+1} \rangle$

at time 1, $\vec{v}(1) = \langle \ln(2) \cdot 2^1, 2 \cdot 1, \frac{1}{1+1} \rangle = \boxed{\langle 2\ln(2), 2, \frac{1}{2} \rangle}$

$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \langle \ln(2)^2 e^{\ln(2)t}, 2, -\frac{1}{(1+t)^2} \rangle$

$\vec{a}(t) = \langle \ln(2)^2 \cdot 2^t, 2, -(1+t)^{-2} \rangle$ @ $t=1$ $\boxed{\vec{a}(1) = \langle 2\ln(2)^2, 2, -\frac{1}{4} \rangle}$

Ex: Find velocity and position functions given the curve with $\vec{a}(t) = \langle \sin(t), 2\cos(t), 6t \rangle$ and $\vec{v}(0) = \langle 0, 0, -1 \rangle$, $\vec{r}(0) = \langle 0, 1, -4 \rangle$

Sol: $\vec{v}(t) = \int \vec{a}(t)\, dt$

$= \langle -\cos(t), 2\sin(t), 3t^2 \rangle + \vec{c}$ Now $\langle 0, 0, -1 \rangle = \vec{v}(0) = \langle -\cos(0), 2\sin(0), 3(0)^2 \rangle + \vec{c}$

$= \langle -1, 0, 0 \rangle + \vec{c}$

$\therefore \vec{c} = \langle 0, 0, -1 \rangle - \langle -1, 0, 0 \rangle = \langle 1, 0, -1 \rangle$

so $\vec{v}(t) = \langle -\cos(t), 2\sin(t), 3t^2 \rangle + \langle 1, 0, -1 \rangle$

$= \langle -\cos(t) + 1, 2\sin(t), 3t^2 - 1 \rangle$

Now $\vec{r}(t) = \int \vec{v}(t)\, dt$

$= \langle t - \sin(t), -2\cos(t), t^3 - t \rangle + \vec{d}$

so $\vec{d} = \vec{r}(0) - \langle 0 - \sin(0), -2\cos(0), 0^3 - 0 \rangle = \langle 0, 1, -4 \rangle - \langle 0, -2, 0 \rangle = \langle 0, 3, -4 \rangle$

$\vec{r}(t) = \langle t - \sin(t), -2\cos(t) + 3, t^3 - t - 4 \rangle$

Ex. When is the speed of particle position trajectory
$\vec{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$ at a min

Sol: The speed function is $f(t) = |r'(t)|$

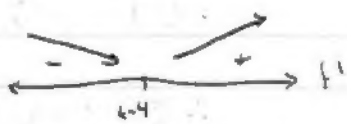
$r'(t) = \langle 2t, 5, 2t - 16 \rangle$ $\quad f(t) = \sqrt{(2t)^2 + 5^2 + (2t-16)^2}$
$f(t) = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256}$ $\quad = (8t^2 - 64t + 281)^{1/2}$

$$\therefore f'(t) = \tfrac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$$
$$= \frac{8t - 32}{(8t^2 - 64t + 281)^{1/2}}$$

Note $64^2 - 4 \cdot 8 \cdot 281 = 2^{12} - 2^5 \cdot 281 < 2^{12} - 2^5 \cdot 256$
$= 2^{12} - 2^5 \cdot 2^8$
$= 2^{12} - 2^{13} < 0$

$\therefore 8t^2 - 64t + 281 = 0$ has no real solutions
the only critical point of this function is at $8t - 32 = 0$ i.e. $t = 4$

Now apply the first derivative test, if $f'(t) < 0$ on $t < 4$ and $f'(t) > 0$ on $t > 4$, then ~~at~~ $t = 4$ corresponds to a minimum



Now $f'(0) = \frac{-32}{\sqrt{281}} < 0$ and $f'(5) = \frac{8}{t} > 0$
hence the particle is slowest @ $t = 4$

Recall: If $f(t) \geq 0$ for all t and is diff. for all t, then f is minimized exactly when $(f(t))^2$ is minimized

Alt. Solution: $f(t) = |r'(t)| = (8t^2 - 64 \cdot 281)^{1/2}$ as before
now minimize $(f(t))^2$

Ex. A ball is thrown with angle $60°$ above ground. If it lands $90\,m$ away, at what speed was it thrown $\quad a = 9.8$

Sol.
$$\begin{cases} a(t) = \langle 0, -9.8 \rangle = \langle 0, -\frac{49}{5} \rangle \\ \vec{V}(0) = |\vec{V}(0)| \langle \cos \pi/3, \sin \pi/3 \rangle = c \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle \\ \vec{r}(t_0) = \langle 90, 0 \rangle \end{cases}$$

want: $|\vec{V}(0)| = c$

$$\therefore \vec{V}(t) = \int \vec{a}(t)\,dt = \langle \alpha, -\frac{49}{5}t + \beta \rangle$$

$$V(0) = \frac{c}{2} \langle 1, \sqrt{3} \rangle$$

$$\therefore \begin{cases} \alpha = \frac{c}{2} \\ \beta = \frac{\sqrt{3}c}{2} \end{cases} \qquad \vec{V}(t) = \langle \frac{c}{2}, -\frac{49}{5}t + \frac{\sqrt{3}}{2}c \rangle$$

$$\vec{r}(t) = \int V(t)\,dt = \langle \frac{c}{2}t + \gamma, -\frac{49}{10}t^2 + \frac{\sqrt{3}}{2}ct + \delta \rangle$$

Now at some time t_0 we have

$$\vec{r}(t_0) = \langle 90, 0 \rangle = \langle \frac{c}{2}t_0 + \gamma, -\frac{49}{10}t^2 + \frac{\sqrt{3}}{2}ct_0 + \delta \rangle$$

Note: with assumption $\vec{r}(0) = \langle 0, 0 \rangle$, we obtain $(\gamma, \delta) = 0$

$$\therefore \vec{r}(t_0) = \langle 90, 0 \rangle = \langle \frac{c}{2}t_0, -\frac{49}{10}t_0^2 + \frac{\sqrt{3}}{2}ct_0 + \delta \rangle \qquad \therefore \frac{c}{2}t_0 = 90 \quad t_0 = \frac{180}{c}$$